Solvable and Triangularizable Groups

Jesse Vogel

based on Chapter 17 of Algebraic groups by J. S. Milne

– Triangularizable groups –

Definition 1. An algebraic group G is **triangularizable** (or **trigonizable**) if every non-zero representation of G contains a one-dimensional subrepresentation.

Remark 2. On the level of comodules (V, ρ), this means there exists a non-zero vector $v \in V$ with $\rho(v) = a \otimes v$ for some $a \in \mathcal{O}_G(G)$.

Example 3. Examples of triangularizable groups are

- Unipotent groups: they have the stronger condition that there exists a $v \in V$ with $\rho(v) = 1 \otimes v$.
- Diagonalizable groups: representations are direct sums of one-dimensional representations.
- More generally, any subgroup of the upper triangular matrices $\mathbb{T}_n$.

Example 4. The group $G = D_4 \simeq \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}$ is not triangularizable. Consider the natural representation of G on $V = k^2$, then the given matrices have no common eigenvectors, so V has no invariant subrepresentation.

Triangularizable groups can be characterized in a number of ways.

Proposition 5. *Let G be an algebraic group. The following are equivalent.*

(i) G is triangularizable.

(ii) For every representation (V, r) of G, there exists a basis of V for which $r(G) \subset \mathbb{T}_n$, with $n = \dim V$.

(iii) G is isomorphic to an algebraic subgroup of $\mathbb{T}_n$ for some n.

(iv) There exists a normal unipotent algebraic subgroup U of G such that G/U is diagonalizable.

Proof. $(i \Rightarrow ii)$ Proof by induction on $n = \dim V$, where the case $n = 0$ is trivial. For $n > 0$, pick $e_1 \in V$ such that $\langle e_1 \rangle$ is a subrepresentation of V. The induction hypothesis on $V/\langle e_1 \rangle$ gives a basis $\overline{e}_2, \ldots, \overline{e}_n$ for $V/\langle e_1 \rangle$ such that G acts via $\mathbb{T}_{n-1}$. Lifting each $\overline{e}_i$ to some $e_i \in V$ gives a basis $e_1, \ldots, e_n$ of V such that G acts via $\mathbb{T}_n$.

$(ii \Rightarrow iii)$ Apply (ii) to a faithful finite-dimensional representation of G.

$(iii \Rightarrow iv)$ Embed $G \subset \mathbb{T}_n$ and let $U = G \cap \mathbb{U}_n$. Then $U \subset G$ is normal and unipotent since $\mathbb{U}_n \subset \mathbb{T}_n$ is. Now G/U injects into $\mathbb{T}_n/\mathbb{U}_n \simeq \mathbb{G}_m^n$, and a subgroup of a diagonalizable group is diagonalizable.

$(iv \Rightarrow i)$ Take some $U \subset G$ as in (iv) and a representation (V, r) of G. Since U is unipotent, we have $V^U \neq 0$, and since U is normal in G, we have that V^U is stable under G. Namely, for any $v \in V^U$ with $n \in U(k)$ and $g \in G(k)$ we find that

$$r(n)r(g)v = r(ng)v = r(gn')v = r(g)r(n')v = r(g)v,$$

with $n' = g^{-1}ng \in U(k)$, so $r(g)v \in V^U$ as well. Hence G/U, which is diagonalizable, acts on V^U, so $V^U \neq 0$ is the sum of 1-dimensional subrepresentations. In particular, there exists a one-dimensional subrepresentation of V. □

Corollary 6. *Triangularizable groups are stable under base change, using characterization* (iii)*.*

Proposition 7. *Let G be an algebraic group over k such that G_ℓ is triangularizable, with ℓ/k a separable extension. Then G contains a unique normal unipotent algebraic subgroup G_u such that G/G_u is of multiplicative type. Moreover, G_u contains all unipotent algebraic subgroups of G.*

Proof. If $U \subset G$ is a normal unipotent subgroup with G/U of multiplicative type, then it contains all unipotent $V \subset G$, as the composition $V \to G \to G/U$ is trivial by [1, 15.17]. This shows the uniqueness of U. To show existence, we can assume ℓ/k is a Galois extension, and use characterization (iv) of Proposition 5 to obtain some normal unipotent $U \subset G_\ell$, which being unique, is stable under the Galois group $\mathrm{Gal}(\ell/k)$. Hence, it arises from some subgroup $G_u \subset G$. Now G_u is unipotent since $U = (G_u)_\ell$ is [1, 15.9], and G/G_u is of multiplicative type since $(G/G_u)_\ell = G_\ell/U$ is diagonalizable. □

Proposition 8. *Let V be a finite-dimensional vector space over an algebraically closed field k, and $G \subset \mathrm{GL}(V)$ a smooth commutative algebraic subgroup. Then there exists a basis for V such that $G \subset \mathbb{T}_n$, where $n = \dim V$.*

Proof. From linear algebra we know that any set of commuting matrices can simultaneously be brought to upper triangular form, that is, we can choose a basis for V such that $G(k) \subset \mathbb{T}_n(k)$. Now $G \cap \mathbb{T}_n$ is an algebraic subgroup of G with $(G \cap \mathbb{T}_n)(k) = G(k)$, and since k-points are dense, we have $G \cap \mathbb{T}_n = G$, so that $G \subset \mathbb{T}_n$. □

Corollary 9. *Every smooth commutative algebraic group G over an algebraically closed field k is triangularizable. Namely, apply the above proposition to a faithful finite-dimensional representation of G.*

– Solvable algebraic groups –

Definition 10. An algebraic group G is **solvable** if there exists a series of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_t = 1$$

such that G_{i+1} is normal in G_i, and each quotient G_i/G_{i+1} is commutative.

Theorem 11 (Lie-Kolchin)**.** *Let G be an solvable, smooth, connected algebraic group over an algebraically closed field k. Then G is triangularizable.*

Proof. Proof by induction on the dimension of G. It suffices to show that any simple representation (V, r) of G is one-dimensional.

Let $N = [G, G]$ be the derived subgroup of G, which is a smooth, connected and normal, and since G is solvable, $\dim N < \dim G$. By the induction hypothesis, there exists a one-dimensional subrepresentation of V when restricted to N. In particular, $V_\chi \neq 0$ for some character $\chi : N \to \mathbb{G}_m$. Let W be the sum of all such V_χ, then by [1, 4.17], this sum is direct $W = \bigoplus_\chi V_\chi$, and so the set S of characters of N for which $V_\chi \neq 0$, is finite. Note that

$$r(n)r(g)x = r(gg^{-1}ng)x = r(g)\chi(g^{-1}ng)x = \chi(g^{-1}ng)r(g)x,$$

for all $x \in V_\chi$, $g \in G(k)$ and $n \in N(k)$, so we find that $G(k)$ permutes the set S.

Now take any $\chi \in S$ and let $H \subset G(k)$ be the stabilizer of χ, that is, $H = \{g \in G(k) : \chi(n) = \chi(g^{-1}ng) \text{ for all } n \in N(k)\}$. Then H is closed in G, since it is given by a set of equations, and H is of finite index, since S is a finite set. But that implies that the cosets of H define a partition of $G(k)$, and since G is connected, we must have $H = G(k)$. Therefore, $G(k)$ stabilizes V_χ, and so does G. As V is simple, we must have $V = V_\chi$.

Now any $n \in N(k)$ is a product of commutators, so it acts on V via an automorphism of determinant 1. The determinant of $x \mapsto \chi(n)x$ is $\chi(n)^{\dim V}$, so χ factors as $\chi : N \to \mu_{\dim V} \to \mathbb{G}_m$. But since N is smooth and connected, χ must be trivial. Hence, G acts through G/N on V. But G/N is commutative, and the result follows from Corollary 9. □

Example 12. The conditions in the above theorem are all necessary:

- (*connected*) The algebraic group $G = D_4$ as in Example 4 is smooth, solvable but not connected, and indeed we have seen that it is not triangularizable.
- (*smooth*) Let k be an (algebraically closed) field of characteristic 2, and let $G \subset \mathrm{SL}_2$ be the algebraic group given by

  $$G(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R) : a^2 = d^2 = 1 \text{ and } b^2 = c^2 = 0 \right\}.$$

  Then G is connected (topologically it is a point) but not smooth, and the exact sequence

  $$1 \to \mu_2 \xrightarrow{a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)} G \xrightarrow{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (ab, cd)} \alpha_2 \times \alpha_2 \to 1$$

  shows that G is solvable. However, G is not triangularizable since the natural action of G on k^2 does not fix any line.
- (*solvable*) Any algebraic subgroup of $\mathbb{T}_n$ is solvable.
- (*algebraically closed*) Let $k = \mathbb{R}$ and consider the algebraic group G given by

  $$G(R) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(R) : a^2 + b^2 = 1 \right\}.$$

  Then G is solvable (since it is commutative), connected and smooth. However it is not triangularizable since it does not have eigenvectors when acting naturally on $\mathbb{R}^2$.

Theorem 13. *Let G be a triangularizable algebraic group. Then there exists a normal series*

$$G \supset G_0 \supset G_1 \supset \cdots \supset G_r = 1$$

such that $G_0 = G_u$ (see Proposition 7), and the action of G on G_i/G_{i+1} by conjugation factors through G/G_u. Moreover, there exists an embedding $G_i/G_{i+1} \to \mathbb{G}_a$ and the action extends to a linear action of G/G_u on $\mathbb{G}_a$.

Proof. Recall the normal series for $\mathbb{U}_n$, given by

$$\mathbb{U}_n = U_0 \supset U_1 \supset \cdots \supset U_{n(n-1)/2} = 1,$$

where

$$U_r(R) = \{(a_{ij}) \in \mathbb{U}_n(R) : a_{ij} = 0 \text{ for } (i,j) = C_\ell \text{ with } \ell \leq r\}.$$

We have that $U_i/U_{i+1} \simeq \mathbb{G}_a$, and $\mathbb{T}_n$ acts linearly (by conjugation) on U_i/U_{i+1}, and this action factors through $\mathbb{T}_n/\mathbb{U}_n$.

For G, use Proposition 5 (iii) to embed $G \subset \mathbb{T}_n$. Then the exact sequence

$$1 \to \mathbb{U}_n \to \mathbb{T}_n \xrightarrow{q} \mathbb{D}_n \to 1$$

yields

$$1 \to G \cap \mathbb{U}_n \to G \to q(G) \to 1.$$

Note that $G \cap \mathbb{U}_n = G_u$, since any unipotent $U \subset G$ has that $q(U) \subset \mathbb{D}_n$ is unipotent and diagonalizable, so trivial, so $U \subset G \cap \mathbb{U}_n$. Now take $G_i = G \cap U_i$. Then G_{i+1} is normal in G_i and G_i/G_{i+1} is a subgroup of $U_i/U_{i+1} \simeq \mathbb{G}_a$. Indeed the group G acts on each G_i/G_{i+1} through an action that extends to $\mathbb{G}_a$, namely the one coming from U_i/U_{i+1}. Note action of G_u is trivial (one-dimensional unipotent actions must be trivial by definition), so the action factors through G/G_u. □

Theorem 14. *Let G be a triangularizable algebraic group over an algebraically closed field k. Then the exact sequence $1 \to G_u \to G \xrightarrow{q} D \to 1$ splits.*

Proof. Proof by induction on the length of the normal series of G from Theorem 13. If this length is zero, then G_u is trivial and we are done. Otherwise, let $N = G_i$ be the last non-trivial group in the subnormal series. Then G/N is still triangularizable, and we have a sequence

$$1 \to G_u/N = (G/N)_u \to G/N \xrightarrow{\overline{q}} D \to 1.$$

By the induction hypothesis, the statement holds for G/N, so let $\overline{s} : D \to G/N$ be a section for $\overline{q}$. Now consider the extension by pullback:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & G' & \xrightarrow{q'} & D & \longrightarrow & 1 \\
 & & \| & & \downarrow h & & \downarrow \overline{s} & & \\
1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{p} & G/N & \longrightarrow & 1
\end{array}$$

Since $N \subset U_i/U_{i+1} \simeq \mathbb{G}_a$ and $D \simeq G/G_u$ acts linearly on $\mathbb{G}_a$, it follows from [1, 16.41d] that $\mathrm{Ext}^1(D, N) = 0$, so the top extension splits. Hence, there exists a section $s' : D \to G'$, and we define $s = h \circ s'$. Indeed, s is now a section for q as $qs = \overline{q}phs' = \overline{q}\overline{s}q's' = \mathrm{id}$. □

Theorem 15. *Let $1 \to U \to G \to D \to 1$ be an extension of a diagonalizable group D by a unipotent group U over an algebraically closed field k. Any two sections $s_1, s_2 : D \to G$ (as group morphisms) differ by an inner automorphism $\mathrm{inn}(u)$ for some $u \in U(k)$.*

Proof. Note that G is triangularizable by Proposition 5 (iv). Proof by induction on the length of the normal series of G from Theorem 13. Both $\overline{s}_1 := p \circ s_1$ and $\overline{s}_2 := p \circ s_2$ (with $p : G \to G/N$) are sections of the sequence $1 \to U/N \to G/N \to D \to 1$. Similarly to the proof of Theorem 14, construct the extension by pullback:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & G' & \xrightarrow{q'} & D & \longrightarrow & 1 \\
 & & \Vert & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overline{s}_1} & & \\
1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{p} & G/N & \longrightarrow & 1
\end{array}$$

By the induction hypothesis, there exists a $\overline{u} \in (U/N)(k)$ such that $\mathrm{inn}(\overline{u}) \circ p \circ s_2 = \overline{s_1}$. Let $u \in U(k)$ be a lift of $\overline{u}$, then $p \circ \mathrm{inn}(u) \circ s_2 = \overline{s}_1$. Since G' is a pullback, we have sections $\sigma_1, \sigma_2 : D \to G'$ such that $s_1 = h \circ \sigma_1$ and $\mathrm{inn}(u) \circ s_2 = h \circ \sigma_2$. Since N is commutative unipotent and D is diagonalizable, it follows from [1, 16.3] that $\sigma_2 = \mathrm{inn}(n) \circ \sigma_1$ for some $n \in N(k)$, and hence $\mathrm{inn}(n) \circ s_1 = \mathrm{inn}(u) \circ s_2$. □

Theorem 16. *Let G be a triangularizable algebraic group over an algebraically closed field k. The maximal diagonalizable subgroups of G are of the form $s(D)$ with $s : D \to G$ a section of $1 \to G_u \to G \to D \to 1$, and any two such maximal subgroups are conjugate by an element of $G_u(k)$.*

Proof. We already know from Theorem 15 that two sections differ by an inner automorphism, which implies the last statement.

Let S be a diagonalizable subgroup of G. Then $S \cap G_u = 1$ since any diagonalizable unipotent group is trivial, so $q : S \xrightarrow{\sim} q(S)$. Let $G' = q^{-1}(q(S))$ and $q' = q|_{G}$, that is, we have an extension by pullback:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_u & \longrightarrow & G' & \xrightarrow{q'} & q(S) & \longrightarrow & 1 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_u & \longrightarrow & G & \xrightarrow{q} & D & \longrightarrow & 1
\end{array}$$

Now, whenever $s : D \to G$ is a section of the bottom (such exist by Theorem 14), the top sequence is split by $s' = s|_{q(S)}$. Since S is a section of q', there exists by Theorem 15 a $u \in G_u(k)$ with $S = (\mathrm{inn}(u) \circ s' \circ q)(S)$, and thus $S \subset \mathrm{inn}(u)s(G/G_u \simeq D)$. Hence $s(G/G_u \simeq D)$ is a maximal diagonalizable subgroup of G, and all such are conjugate. □

References

[1] J.S. Milne, Algebraic Groups